Tangent Complex

Recap What we have vs. What we want

$$\begin{tikzcd}
\mathrm{Lie} \arrow[rr, "\Phi"] \arrow[dr, "C^*_{\mathrm{Lie}}"'] \arrow[ddr, bend right, "\text{forget}"'] & & \mathrm{FMP} \arrow[ll, bend right=60, dashed] \arrow[ddl, bend left, "?"] \\
& (\mathrm{CAlg}^{\mathrm{aug}})^{\mathrm{op}} \arrow[ul, bend right, "\mathfrak{D}"'] \arrow[ur, "\mathrm{Spf}"'] & \\
& \mathrm{Vect}^{\mathrm{dg}} &
\end{tikzcd}$$

$\mathfrak{D}$ = Koszul duality functor producing a Lie algebra from an augmented commutative algebra

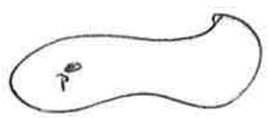
__Question__: Given a formal moduli problem (or augmented algebra), how should we "linearize" to get a dg vector space?

__Want__: the linearization of $C^*_{\mathrm{Lie}}(\mathfrak{g})$ should return $\mathfrak{g}$ (or something equivalent)

[Solicit guesses]

Let's take a straightforward approach from (algebraic) geometry:

$$A \xrightarrow{\varepsilon} k \quad \leftrightarrow \quad \operatorname{Spec} k \xrightarrow{p} \operatorname{Spec} A$$



$\Rightarrow$ take Zariski tangent space

$$T_p^* \operatorname{Spec} A := m_\varepsilon / m_\varepsilon^2 , \quad m_\varepsilon = \ker(\varepsilon)$$

$$T_p \operatorname{Spec} A = \operatorname{Hom}_k (T_p^* \operatorname{Spec} A, k)$$

Let's try $A = C^*_{Lie}(g) \xrightarrow{\varepsilon} \operatorname{Sym}^0 \cong k$

$$\Rightarrow m_\varepsilon = \operatorname{Sym}_k^{\geq 1}(g^*[-1]) = (g^*[-1])$$

$$\Rightarrow m_\varepsilon^2 = \operatorname{Sym}^{\geq 2}(g^*[-1])$$

$$\Rightarrow T_\varepsilon^* \cong g^*[-1]$$

$$\Rightarrow T_\varepsilon \cong g[1]$$

Looks good! We get back g, up to a shift

Ansatz: linearize by taking the "tangent complex" of $X \in FMP$ at its "base point" $X(k)$

dg vector space

Remark: You might worry that our computation was too naive. We should take the homotopy fiber (= kernel) instead of just m & the homotopy cofiber (= cokernel) instead of just m/m^2.

If so, I'm glad you're worrying!

Here is why the naive construction is homotopically correct here.

Fact: Given a diagram $A \to B \leftarrow C$, if A, B, C are fibrant & at least one map is a fibration, then the naive pullback $A \times_B C$ is a homotopy pullback. Simplifies computations (cf last week)

(Dually for $A \leftarrow B \to C$, w/ all cofibrant & one map a cofibration).

For us, every algebra is fibrant & $\varepsilon : C^*_{Lie} \mathfrak{g} \to k$ is a fibration, etc.

Let's try now to apply our ansatz of taking the tangent space.

Recall that there's another way to understand the tangent space using functor of points:

For $p : \operatorname{Spec} k \longrightarrow \operatorname{Spec} A$,

$$T_p = \left\{ \begin{tikzcd} \operatorname{Spec} k \arrow[r, "p"] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \arrow[r] \arrow[ur, dashed, "\tilde{p}"] & \operatorname{Spec} k \end{tikzcd} \right\} \quad \text{lifts}$$

$$= \left\{ \begin{tikzcd} k \arrow[r, "i"] \arrow[d] & A \arrow[d, "\varepsilon"] \arrow[dl, dashed, "\tilde{\varepsilon}"'] \\ k[\varepsilon]/\varepsilon^2 \arrow[r] & k \end{tikzcd} \right\}$$

extensions

$$\begin{tikzcd} k \cdot \varepsilon = (\varepsilon) \arrow[d] & m_A \arrow[l] \arrow[d] \\ k[\varepsilon]/\varepsilon^2 \arrow[d] & A \arrow[l, "\tilde{\varepsilon}"'] \arrow[d, "\varepsilon"] \\ k \arrow[r, equal] & k \end{tikzcd}$$

in fact, all such maps are given by

$$\left\{ m_A / m_A^2 \longrightarrow (\varepsilon) \cong k \right\}$$

This approach generalizes nicely: degree zero

we will replace the dual numbers $k[\varepsilon]/(\varepsilon^2)$ by

the <u>shifted</u> dual numbers

$$\{ k[\varepsilon_{-n}] / (\varepsilon_{-n}^2) \}_{n \geq 0}$$

and study lifts, i.e.,

$$\{ X (k[\varepsilon_{-n}]/(\varepsilon_{-n}^2)) \}_{n \geq 0}$$

This sequence of spaces forms the homotopical analog of a vector space: it is a

$$\underbrace{k\text{-module}}_{\substack{\text{homotopical}\\ \text{analog of}\\ k\text{-module}}} \quad \underbrace{\text{spectrum}}_{\substack{\text{homotopical analog of}\\ \text{abelian group}}}$$

<u>Fact</u>: Such a spectrum is equivalent to a dg vector space

It is called the <u>tangent complex</u> (or spectrum) of X.

We will now explain what we've just said.

Spectra

I wish I had a better way to motivate. Instead, I'll just be efficient & abstract.

Def Let $\mathcal{S}_*$ denote the ∞-category of pointed spaces, i.e., an object is an arrow $* \xrightarrow{f} X$.

Let $\mathcal{S}_*^{fin}$ denote the smallest full subcategory containing the terminal object $* \to *$ & closed under finite colimits.

Remarks

1) You can construct $\mathcal{S}_*^{fin}$ by taking the collection of all pointed simplicial sets X such that X has only finitely many nondegenerate simplices:

$$sSet_*^{fin} \hookrightarrow sSet_*$$

functors out of them are determined by finite quantity of data

and considering the associated sub ∞-category.

2) Topologically, you can think of finite cell complexes (w/ a base point)

3) The "spheres" generate under colimits (e.g., just a nondegenerate 0- & n-simplex)

there is an initial & terminal object & they are equivalent

Def For $\mathcal{C}$ a pointed ∞-category, a <u>spectrum object in $\mathcal{C}$</u> is a functor $F: \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$ such that

1) $F(*) \simeq 0_{\mathcal{C}}$ "reduced"

2) F sends a pushout square to a pullback square "excisive" "Mayer-Vietoris"

These conditions ensure that if $\mathcal{C}$ possesses finite colimits, then F is determined by its value on spheres, since they generate $\mathcal{S}_*^{\mathrm{fin}}$.

Prop Let $\mathcal{C}$ be pointed & possess finite colimits. A spectrum $F: \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{C}$ is determined by a sequence of objects $F_E = \{E_n\}_{n \in \mathbb{N}}$ in $\mathcal{C}$ and equivalences

$$E_n \simeq \Omega E_{n+1} := 0 \times_{E_{n+1}} 0$$

Pf Given F, define $E_n := F(S^n)$.

Now $S^{n+1} \simeq \Sigma S^n := * \sqcup_{S^n} *$ use homotopy pushout in top. spaces but as F is excisive, we see

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.O.}"] & * \arrow[d] \\ * \arrow[r] & S^{n+1} \end{tikzcd} \implies \begin{tikzcd} E_n \simeq F(S^n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & F(*) \simeq 0_C \arrow[d] \\ 0_C \simeq F(*) \arrow[r] & F(S^{n+1}) \simeq E_{n+1} \end{tikzcd}$$

So $E_n \simeq 0_C \times_{E_{n+1}} 0_C \simeq \Omega E_{n+1}$

Conversely, every finite space is obtained by gluing cells, i.e., by a sequence of pushouts using spheres. Hence, given the data

$$\{E_n\}_{n \in \mathbb{N}} \ \& \ \{E_n \simeq \Omega E_{n+1}\},$$

the value of F is determined uniquely up to a contractible space of choices □

Remarks:

The classic example is a homology theory

$$H^*: S_*^{fin} \longrightarrow dgVect_k \text{ or } Ch_{\mathbb{Z}} \ldots$$

where $H^*(*) \simeq 0$ & H^* satisfies excision

This is a kind of "linearization" of spaces.

Let's apply this in our context.

<u>Prop</u> The sequence $\{k[\varepsilon_{-n}]/(\varepsilon_{-n}^2)\}_{n\geq 0}$ is a spectrum in $\mathrm{CAlg}_k^{\mathrm{aug}}$

<u>Pf</u> We need to show that

$$k \oplus k\varepsilon_{-n} \simeq \Omega(k \oplus k\varepsilon_{-n-1}) = k \times_{k\oplus k\varepsilon_{-n-1}} k$$

We thus compute the homotopy pullback:

replacement by fibration

$$\begin{tikzcd} & k \arrow[d] \arrow[r, phantom, "\simeq"] & k \oplus (k\varepsilon_{-n-1} \xrightarrow{id} k\varepsilon_{-n}) \arrow[dl] \\ k \arrow[r] & k\oplus k\varepsilon_{-n-1} & \end{tikzcd}$$

$$0$$

so pullback is $k \oplus k\varepsilon_{-n}$. $\square$

<u>Lemma</u> This spectrum factors through the <u>small</u> algebras

$$E : \mathcal{S}_+^{\mathrm{fin}} \longrightarrow \mathrm{CAlg}^{\mathrm{sm}} \hookrightarrow \mathrm{CAlg}^{\mathrm{aug}}.$$

<u>Proof sketch</u>: ~~Any finite space $K \in \mathcal{S}_+^{\mathrm{fin}}$ is a sequence of pushouts with spheres from the set $\pi_0(K)$. That~~

~~[illegible]~~ Proof

Let $K \in S_*^{\mathrm{fin}}$. Then there is a sequence of "cell attachments"

$$K = K_0 \to K_1 \to \cdots \to K_n = \pi_0 K$$

where each $K_i \to K_{i+1}$ sits in a pushout

$$\begin{array}{ccc} K_i & \longrightarrow & * \\ \downarrow & & \downarrow \\ K_{i+1} & \longrightarrow & S^n \end{array}$$

Then E gives us a pullback

$$\begin{array}{ccc} E(K_i) & \longrightarrow & k \\ \downarrow & & \downarrow \\ E(K_{i+1}) & \longrightarrow & k \oplus k\varepsilon_{-n} \end{array}$$

So $E(K_i) \to E(K_{i+1})$ is an elementary morphism. $\square$

We now introduce

<u>Def</u> The composite functor

$$X \circ E : S_*^{\mathrm{fin}} \longrightarrow S$$

is the <u>tangent complex</u> for $X \in \mathrm{FMP}$.

<u>Claim</u> The tangent complex is reduced & excisive. Hence it defines a <u>spectrum</u> in S.

[Boring proof]

~~[illegible]~~

Theorem

Let $f: X \to Y$ be a map of formal moduli problems.

f is an equivalence $\Longleftrightarrow$ $f \circ E$ is an equivalence of tangent complexes

Pf $\Rightarrow$: obvious

$\Leftarrow$: For $A \in \mathrm{CAlg}^{sm}$, pick a sequence of elementary extensions in CAlg^{sm}:

$$A = A_n \to A_{n-1} \to \cdots \to A_0 \simeq k.$$

~~We~~ use induction. For $i=0$, $\underset{*}{\underset{\parallel}{X(k)}} \xrightarrow{f(k)} \underset{*}{\underset{\parallel}{Y(k)}}$ $\simeq$.

Consider induction step:

$$\begin{tikzcd} A_{i+1} \arrow[r] \arrow[d] & k \arrow[d] \\ A_i \arrow[r] & k \oplus k\varepsilon_{-d} \end{tikzcd}$$

$X(k) \simeq Y(k) = *$

$\Rightarrow$ get a fiber sequence

$$\begin{tikzcd} X(A_{i+1}) \arrow[r] \arrow[d, "f(A_{i+1})"'] & X(A_i) \arrow[r] \arrow[d, "f(A_i)"', "\simeq \text{ by hypothesis}"] & X(k \oplus k\varepsilon_{-d}) \arrow[d, "f(\varepsilon_d)"', "\simeq \text{ by hypothesis}"] \\ Y(A_{i+1}) \arrow[r] & Y(A_i) \arrow[r] & Y(k \oplus k\varepsilon_{-d}) \end{tikzcd}$$

Now take the associated map btw long exact sequences of homotopy groups. We see that

$\pi_n(f(A_{i+1}))$ is an equivalence $\forall n$ so

$f(A_{i+1})$ is a weak equivalence of spaces. $\square$

This fact is <u>amazing</u>. It tells us we can identify two formal moduli problems as equivalent just by testing on the shifted dual numbers! (Not just values: need a map btw them!)

Thus

$$\begin{aligned} T: \mathrm{FMP} &\to \mathrm{dgVect} \\ X &\mapsto X \circ E \end{aligned}$$

<u>detects</u> equivalences & preserves limits,

just like

$$\mathrm{forget}: \mathrm{Lie} \to \mathrm{dgVect}$$